PhD proposal

Orthotropic maps for mesh generation

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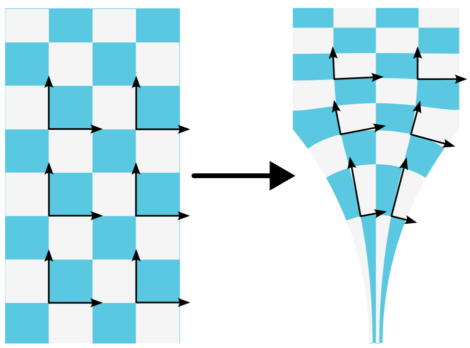
I. Context

Computing planar maps is a very old problem that has occupied geographers and mathematicians for centuries in order to best describe the earth at different scales. To compute a map, two objectives come into play: it must be injective (meaning distinct points on earth correspond to distinct points on the map) and it must faithfully represent the geometry (that is to say the map distortion is minimal). In a modern version of this problem, maps are used to describe complex surfaces or volumes stored as meshes in a computer. Discrete geometries are mapped to simpler domains (such as plane or sphere) for texture mapping, but more complex target domains are also considered. Of particular relevance to us are atlases, which consist of collections of maps with specific constraints along cuts. These atlases are typically used for quadrilateral or hexahedral mesh generation [1].

Maps are commonly computed using numerical optimization by directly moving the mesh vertices to minimize a given distortion measure subject to an injectivity constraint. This measure is often split in two parts: the distortion of areas and the distortion of angles. Local preservation of area is a very weak requirement as a ball can be arbitrarily distorted into any domain with same volume. Consequently, this property alone is rarely used in practical scenarios. On the contrary, angle-preserving maps, also called *conformal*, have a long and successful history in map computation for at least two reasons. First, it creates maps without shearing and preserves simple structures like circles, thus it is well-suited for texture mapping. Second, their governing equations are linear and depend on only one function: the scale factor, which represents the ratio between the initial and the deformed area at each point. Moreover, in the continuous setting, injectivity is guaranteed without the need for an additional non-linear energy.

However, besides their undeniable benefits, conformal maps have their own shortcomings. The main one is that they do not generalize well to volumetric maps. Indeed, conformality in 3D is limited to similarities and inversions (see Liouville's theorem [2]) and thus cannot be used for non-trivial tasks. Moreover, for surface mappings, direction constraints can be imposed on boundaries but inner constraints are often out of reach [3]. In this thesis, we would like to preserve the benefits of conformal mappings, such as a small number of parameters, the ease of computation and the absence of complex injectivity penalization, while still being able to compute interesting volumetric maps.

Idea. We propose to study the set of mappings which are entirely free from shear, thus generalizing the concept of conformal maps. Such maps only allow independent stretching in two (or three) orthogonal directions. Inspired by materials science, we refer to them as *"orthotropic"* maps. Intuitively, on a planar domain, we define at each point a reference frame aligned with the global refence system and locally attached to the material. Our degrees of freedom are the rotation of these frames and the independent scaling of the two vectors. More general atlases can be computed by employing discontinuous fields of frames with specific singularity patters [1]. This deformation of the domain transforms infinitesimal squares,



initially aligned with the reference frames, into infinitesimal rectangles. This property is essential in applications such as mesh generation as discussed in Section III.

Challenges. The goal of this thesis is to develop a framework to study and to numerically compute orthotropic mappings with a particular focus on volumetric maps and atlases.

II. Methodology

In order to theoretically study orthotropic deformations, we use Cartan's method of moving frames [4]. The main idea is to define a system of orthogonal frames and scale functions that smoothly evolve in domain of arbitrary dimension and to characterize when they infinitesimally define a valid deformation. More precisely, let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a planar orthotropic map, then its Jacobian must be such that $\nabla f = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix} E^\top$, where $E: \mathbb{R}^2 \to SO(2)$ is a rotation field and $a, b: \mathbb{R}^2 \to \mathbb{R}$ are the scaling functions. The map is then locally characterized by the necessary condition: $\nabla \times \nabla f = 0$. We can rewrite this equation to fully express the infinitesimal rotation, which relates two infinitesimal close rotations E in terms of a, b and E. This necessary (and sufficient in many useful cases) integrability condition is linear with respect to a and b.

To make this theoretical analysis useful for applications, we propose the following tasks:

Task 1: Orthotropic parametrizations of surfaces. The first step is to explore the theoretical foundations of orthotropic deformations for surface texture mapping and quad-remesing. Our goal is to discretize and efficiently solve the non-linear integrability equation. At this stage, we assume that the positions of the singular vertices of the target quad-mesh are given, meaning that the rotation field's singularities are known and constant. Preliminary results illustrating this PhD proposal have already been obtained in collaboration with Keenan Crane (Associate Professor at Carnegie Mellon University).

Task 2: Orthotropic mapping in volumes. The integrability condition for the existence of an orthotropic map can be easily extended to the volumetric case. However, this additional dimension introduces new challenges. 3D rotations are no longer commutative making the optimization significantly more difficult, and as a result, our solver for surface maps may become obsolete. We once again assume that rotation field singularities are given as input. However, the map near a singularity becomes extremely distorted. Since the singularities now stretch along lines, we need to ensure that the discretization has enough degrees of freedom to capture this highly anisotropic effect. Therefore, higher-order elements may need to be considered.

Finally, a more theoretical aspect must be studied: can an orthotropic map incorporate all possible valid singularity graphs? An orthotropic map can contain singularity lines, as it is a simple extrusion of a surface singular map. However, if a singularity graph has a vertex (an intersection of two or more singular lines), is there an orthotropic map with such a singularity configuration?

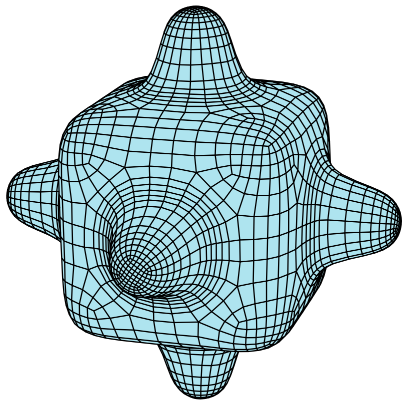
Task 3: Singular mappings. Singular graphs are not well understood in theory. While local obstructions to the existence of a mapping near a singular edge are well known [5], global results are scarce and often very algebraic [6], which makes them challenging to use in practice. Therefore, finding a singularity graph such that boundary constraints are enforced is extremely challenging and beyond the capabilities of state-of-the-art algorithms. The integrability condition of orthotropic maps offers a practical way to find a valid singularity graph by jointly computing the orthotropic scales a, b, c and a rotation field E with potential singularities.

Interestingly, in the case of surfaces, the orthotropic scales a, b are continuous for singularity indices $p/2,\ p \in \mathbb{Z}$, and can be computed through continuous optimization. These singularities are typically created by the principal directions of symmetric matrix and could be very useful when designing lattice material for additive manufacturing. However, they are incompatible with meshing applications that require $p/4$ index singularities. In such cases, the orthotropic scales become discontinuous, making the optimization extremely challenging. The simplest algorithm would be to determine, at each step, if the local frames jump depending on the satisfaction of the orthotropic integrability criterion. This is highly challenging as numerical errors can lead to false positive cases.

Task 4: Beyond orthotropy. Our last step is to go beyond orthotropic deformations and discover which maps can be computed while keeping two key properties: a characterization by a small number of meaningful parameters and convergence under refinement to an injective map. A closer examination of the theory reveals that these properties are maintained if the Jacobian of the map f can be written as the product $\nabla f = AE$ where $A \in \mathbb{A}$ belongs to a commutative matrix Lie group with positive determinant, and $E \in \mathbb{E}$ belongs to any Lie group with positive determinant. In particular, if $\mathbb{E}$ is the special linear group and A is the identity matrix, then the map is area-preserving. Other combinations can be envisioned, opening the door to many other applications.

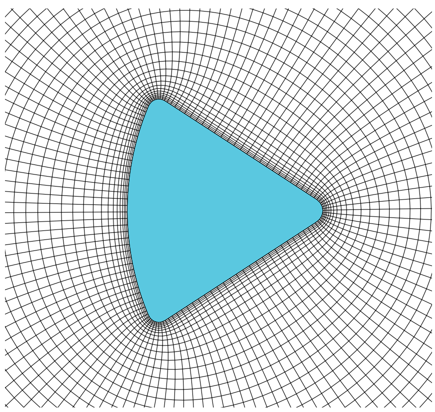
III. Orthotropic map applications for remeshing

Anisotropic remeshing. The objective of quad (or hex) remeshing is to generate a mesh that accurately approximate a target geometry while maintaining a fixed number of elements. When approximating a surface using a quad mesh, theoretical findings indicate that the edges should align with the principal (orthogonal) curvature directions, and the aspect ratio of the elements should be in proportion to the ratio of the principal curvatures [7] as in the inset figure. This result is a quite intuitive because regions with high curvature demand smaller elements for a precise approximation. Similarly, to reduce numerical errors in numerical simulations, the mesh should be denser in areas where the expected solution exhibits significant variations and less dense in areas where the solution is nearly flat. Once again, the theory suggests that the most accurate quad (or hex) mesh must have edges aligned with the (orthogonal) eigenvectors of the function's Hessian [8]. Clearly, both aspects of the approximation problem can be addressed by meshes with rectangular (or rectangular cuboid) elements, which can be extracted from an orthotropic map.



Numerical simulation. It is well-known that the finite element method exhibits improved convergence properties when elements approach perfect squares or cubes, and the convergence is not guaranteed in the presence of non-convex elements [9]. By enforcing rectangular elements through orthotropic mappings, we not only avoid non-convexity but also fulfill the requirements for the optimal convergence of the popular serendipity elements [10].

Our colleagues at CEA conducts numerous numerical simulations using quad and hex meshes, with a particular focus on boundary layer simulations that necessitate highly anisotropic elements to capture extreme physical phenomena in directions normal to the boundary (for example, Apollo 11 entering the atmosphere). In their current workflow, users manually remesh their models according to their specific needs, which is a time-consuming process, often taking weeks for an engineer to obtain the desired mesh. The CEA is actively engaged in research to reduce the user input during mesh generation. Orthotropic maps present a promising solution to address some of these challenges.



IV. Application and starting date

The PhD position starts in September or October 2024. The candidate must hold a master in computer science or in applied mathematics. Typically, a candidate with knowledge in differential geometry or/and finite element method is appreciated. This PhD offers the opportunity to visit (and work with) Franck Ledoux from CEA and Keenan Crane from Carnegie Mellon University. Applications can be sent in either French or English. To apply for the position, please send a CV to etienne.corman@cnrs.fr.

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